

# Approximate bound states of the Dirac equation with some physical quantum potentials

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## Abstract

The approximate analytical solutions of the Dirac equations with the reflectionless-type and Rosen-Morse potentials including the spin-orbit centrifugal (pseudo-centrifugal) term are obtained. Under the conditions of spin and pseudospin (pspin) symmetry concept, we obtain the bound state energy spectra and the corresponding two-component upper- and lower-spinors of the two Dirac particles by means of the Nikiforov-Uvarov (NU) method in closed form. The special cases of the  $s$ -wave  $\kappa = \pm 1$  ( $l = \tilde{l} = 0$ ) Dirac equation and the non-relativistic limit of Dirac equation are briefly studied.

Keywords: Dirac equation, spin and pseudospin symmetry, Nikiforov-Uvarov method.

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## I. INTRODUCTION

When a particle is exposed to a strong potential field, the relativistic effect must be considered which gives the correction for nonrelativistic quantum mechanics. Taking the relativistic effects into account, a spinless particle in a potential field should be described with the Klein-Gordon (KG) equation. The solution of the Dirac equation is considered important in different fields of physics like nuclear and molecular physics [1,2]. Within the framework of the Dirac equation the spin symmetry arises if the magnitude of the spherical attractive scalar potential  $S$  and repulsive vector  $V$  potential are nearly equal such that  $S \sim V$  in the nuclei (*i.e.*, when the difference potential  $\Delta = V - S = C_- = C_\Delta$ , with  $C_\Delta$  is an arbitrary constant). However, the pseudospin (pspin) symmetry occurs if  $S \sim -V$  are nearly equal (*i.e.*, when the sum potential  $\Sigma = V + S = C_+ = C_\Sigma$ , with  $C_\Sigma$  is an arbitrary constant) [3]. The spin symmetry is relevant for mesons [4]. The pspin symmetry concept has been applied to many systems in nuclear physics and related areas [3-7] and used to explain features of deformed nuclei [8], the super-deformation [9] and to establish an effective nuclear shell-model scheme [5,6,10]. Recently, the spin and pspin symmetries have been widely applied on several physical potentials by many authors [11-26]. For example, the Dirac equation has been solved for the deformed generalized Poschl-Teller (PT) potential [27], modified PT potential [28,29], Manning-Rosen (MR) potential [30], well potential [31], modified Rosen-Morse (RM) potential [32] and class of potentials including harmonic oscillator, Morse, Hulthén, Scarf' Eckart, MR, Trigonometric RM potentials and others [33] in the framework of the approximation to the spin-orbit centrifugal term using the proper quantization rule, algebraic methods, Ladder operators and  $su(2)$  algebra.

The exact solutions of the Dirac equation for the exponential-type potentials are possible only for the  $s$ -wave ( $\kappa = \pm 1$  case) when the spin-orbit coupling term will get suppressed [34]. However, for  $l$ -states an approximation scheme has to be used to deal with the spin-orbit centrifugal  $\kappa(\kappa + 1)/r^2$  (pseudo-centrifugal,  $\kappa(\kappa - 1)/r^2$ ) term. In this direction, many works have been done to solve the Dirac equation with large number of potentials to obtain the energy equation and the two-component spinor wave functions [35-42]. It has been concluded that the values of energy spectra may not depend on the spinor structure of the particle [43], *i.e.*, whether one has a spin-1/2 or a spin-0 particle. Also, a spin-1/2 or a spin-0 particle with the same mass and subject to the same scalar  $S$  and vector  $V$  potentials of

equal magnitude, i.e.,  $S = \pm V$  ( $\Delta = \Sigma = 0$  or  $C_{\pm} = 0$ ), will have the same energy spectrum (isospectrality), including both bound and scattering states [43]. It has been shown that for massless particles (or ultrarelativistic particles) the spin- and pspin spectra of Dirac particles are the same for the harmonic oscillator potentials [44].

Recently, we obtained the spin symmetric and pspin bound state solutions of the Dirac equation with the standard RM well potential model [20,45]:

$$V(r) = -V_1 \sec h^2 \alpha r + V_2 \tanh \alpha r, \quad (1)$$

where the coupling constants  $V_1$  and  $V_2$  denote the depth of the potential and  $\alpha$  is the range of the potential that has an inverse of length dimension. We use the computer software MATLAB and plot the potential (1) for three different set of parameters  $V_1$  and  $V_2$ . It is plotted in Fig. 1.

The aim of the present paper is to extend the  $s$ -wave solutions by solving the Dirac equation with some physical potentials given in Ref. [34] in the framework of the Nikiforov-Uvarov (NU) method [46-50] by taking an approximation to deal with the centrifugal (pseudo-centrifugal) potential term [20,51]. The approximation scheme used to deal with the spin-orbit centrifugal barrier  $\kappa(\kappa+1)/r^2$  holds for values of spin-orbit coupling quantum number  $\kappa$  that are not large and vibrations of the small amplitude [51]. In the presence of spin symmetry  $S \sim V$  and pspin symmetry  $S \sim -V$ , we calculate bound state energy eigenvalues and their corresponding upper and lower spinor wave functions. We also show that the spin and pspin symmetry Dirac solutions when  $\Delta = C_-$  and  $\Sigma = C_+$  can be reduced to the exact spin symmetry and pspin symmetry limitation  $\Delta = 0$  and  $\Sigma = 0$ , respectively. These are found to be identical to the KG solution for the  $V = \pm S$  cases. Furthermore, the bound state solutions of the Schrödinger equation are also obtained from the nonrelativistic limit of the Dirac equation if an appropriate mapping of parameters is used.

The paper is organized as follows: Section 2 is mainly devoted to the basic spin and pspin Dirac equation. In sect. 3, the approximate analytical bound state solutions of the  $(3+1)$ -dimensional Dirac equation with the reflectionless-type and the RM potentials are obtained in the presence of the spin and pspin limits using a parametric generalization of the NU method. In sect. 4, special cases like the  $s$ -wave  $\kappa = \pm 1$  ( $l = \tilde{l} = 0$ ) and nonrelativistic limit are studied. Section 5 gives the relevant conclusion.

## II. BASIC SPIN AND PSPIN DIRAC EQUATIONS

The Dirac equation for fermionic massive spin-1/2 particles subject to vector and scalar potentials is [1]

$$[c\boldsymbol{\alpha} \cdot \mathbf{p} + \beta (Mc^2 + S(r)) + V(r) - E] \psi_{n\kappa}(\mathbf{r}) = 0, \quad \psi_{n\kappa}(\mathbf{r}) = \psi(r, \theta, \phi), \quad (2)$$

where  $E$  is the binding relativistic energy of the system,  $M$  is the mass of a particle,  $\mathbf{p} = -i\hbar\nabla$  is the momentum operator, and  $\alpha$  and  $\beta$  are  $4 \times 4$  Dirac matrices [3,12,20]. The spinor wave functions take the form

$$\psi_{n\kappa}(\mathbf{r}) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r) Y_{jm}^l(\theta, \phi) \\ iG_{n\kappa}(r) Y_{jm}^{\tilde{l}}(\theta, \phi) \end{pmatrix}, \quad (3)$$

where  $F_{n\kappa}(r)$  and  $G_{n\kappa}(r)$  are the radial wave functions of the upper- and lower-spinor components, respectively, and  $Y_{jm}^l(\theta, \phi)$  and  $Y_{jm}^{\tilde{l}}(\theta, \phi)$  are the spherical harmonic functions coupled to the total angular momentum  $j$  and its projection  $m$  on the  $z$  axis.

In the presence of spin symmetry ( i.e.,  $\Delta = C_- = C_\Delta$ ), one obtains a second-order differential equation for the upper-spinor component [12,20,52,53]:

$$F_{n\kappa}''(r) - \left( \frac{\kappa(\kappa+1)}{r^2} + A_s^2 + B_s \Sigma \right) F_{n\kappa}(r) = 0, \quad (4)$$

where

$$A_s^2 = \frac{1}{\hbar^2 c^2} [M^2 c^4 - E_{n\kappa}^2 - (Mc^2 - E_{n\kappa}) C_-], \quad B_s = \frac{1}{\hbar^2 c^2} (Mc^2 + E_{n\kappa} - C_-), \quad (5)$$

and  $\kappa(\kappa+1) = l(l+1)$ ,  $\kappa = l$  for  $\kappa < 0$  and  $\kappa = -(l+1)$  for  $\kappa > 0$ . Further, the lower-spinor component can be obtained as

$$G_{n\kappa}(r) = \frac{1}{Mc^2 + E_{n\kappa} - C_-} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) F_{n\kappa}(r), \quad (6)$$

where  $E_{n\kappa} \neq -Mc^2$  when  $C_- = C_\Delta = 0$  (exact spin symmetric case). It means that only positive energy spectrum is permitted.

Overmore, in the presence of pspin symmetry ( i.e.,  $\Sigma = C_+ = C_\Sigma$ ), one obtains a second-order differential equation for the lower-spinor component,

$$G_{n\kappa}''(r) - \left( \frac{\kappa(\kappa-1)}{r^2} + A_{ps}^2 - B_{ps} \Delta \right) G_{n\kappa}(r) = 0, \quad (7)$$

where

$$A_{ps}^2 = \frac{1}{\hbar^2 c^2} [M^2 c^4 - E_{n\kappa}^2 + (Mc^2 + E_{n\kappa}) C_+], \quad B_{ps} = \frac{1}{\hbar^2 c^2} (Mc^2 - E_{n\kappa} + C_+). \quad (8)$$

The upper-spinor component  $F_{n\kappa}(r)$  can be obtained by means of

$$F_{n\kappa}(r) = \frac{1}{Mc^2 - E_{n\kappa} + C_+} \left( \frac{d}{dr} - \frac{\kappa}{r} \right) G_{n\kappa}(r), \quad (9)$$

where  $E_{n\kappa} \neq Mc^2$  when  $C_+ = C_\Sigma = 0$  (exact pspin symmetric case). It means that only negative energy spectrum is allowed for this case. From the above equations, the energy eigenvalues depend on the quantum numbers  $n$  and  $\kappa$ , and also the pseudo-orbital angular quantum number  $\tilde{l}$  according to  $\kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1)$ , which implies that  $j = \tilde{l} \pm 1/2$  are degenerate for  $\tilde{l} \neq 0$ . The quantum condition for bound states demands the finiteness of the solution at infinity and at the origin points.

It is known that Eqs. (4) and (7) can be solved exactly only for the case of  $\kappa = -1$  ( $l = 0$ ) and  $\kappa = 1$  ( $\tilde{l} = 0$ ), respectively when the spin-orbit coupling centrifugal and pseudo-centrifugal terms will get suppressed. In the case of nonzero  $l$  or  $\tilde{l}$  values, we can use the approximation scheme to deal with the spin-orbit centrifugal (pseudo-centrifugal) term when  $\kappa$  is not large and when vibrations of the small amplitude near the minimum point  $r = r_e$  [17,51]

$$\frac{1}{r^2} \approx \frac{1}{r_e^2} \left[ D_0 + D_1 \frac{-\exp(-2\alpha r)}{1 + \exp(-2\alpha r)} + D_2 \left( \frac{-\exp(-2\alpha r)}{1 + \exp(-2\alpha r)} \right)^2 \right], \quad (10)$$

where  $D_i$  is the parameter of coefficients ( $i = 1, 2, 3$ ) given by

$$D_0 = 1 - \left( \frac{1 + \exp(-2\alpha r_e)}{2\alpha r_e} \right)^2 \left( \frac{8\alpha r_e}{1 + \exp(-2\alpha r_e)} - (3 + 2\alpha r_e) \right), \quad (11a)$$

$$D_1 = -2(\exp(2\alpha r_e) + 1) \left[ 3 \left( \frac{1 + \exp(-2\alpha r_e)}{2\alpha r_e} \right) - (3 + 2\alpha r_e) \left( \frac{1 + \exp(-2\alpha r_e)}{2\alpha r_e} \right) \right], \quad (11b)$$

$$D_2 = (\exp(2\alpha r_e) + 1)^2 \left( \frac{1 + \exp(-2\alpha r_e)}{2\alpha r_e} \right)^2 \left( 3 + 2\alpha r_e - \frac{4\alpha r_e}{1 + \exp(-2\alpha r_e)} \right), \quad (11c)$$

and higher order terms are neglected.

### A. Spin symmetric solution

We take the sum potential in Eq. (4) in the form of standard RM well potential (1), i.e.,

$$\Sigma = V(r) = -4V_1 \frac{\exp(-2\alpha r)}{(1 + \exp(-2\alpha r))^2} + V_2 \left( \frac{1 - \exp(-2\alpha r)}{1 + \exp(-2\alpha r)} \right). \quad (12)$$

Upon introducing the new variable  $z(r) = \exp(-2\alpha r)$  and substituting the above sum potential into Eq. (4) which then can be cast into the form

$$F''_{n\kappa}(z) + \frac{(1+z)}{z(1+z)} F'_{n\kappa}(z) + \frac{(-a_2 z^2 + a_1 z - a_0^2)}{z^2(1+z)^2} F_{n\kappa}(z) = 0, \quad (13)$$

where at boundaries we require that  $F_{n\kappa}(0) = F_{n\kappa}(-1) = 0$  and the parameters  $a_i$  ( $i = 0, 1, 2$ ) take the forms:

$$\begin{aligned} a_0 &= \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa+1)}{r_e^2} D_0 + B_s V_2 + A_s^2} > 0, \\ a_1 &= \frac{1}{4\alpha^2} \left( \frac{\kappa(\kappa+1)}{r_e^2} (D_1 - 2D_0) + 4B_s V_1 - 2A_s^2 \right), \\ a_2 &= \frac{1}{4\alpha^2} \left( \frac{\kappa(\kappa+1)}{r_e^2} (D_0 - D_1 + D_2) + A_s^2 - B_s V_2 \right). \end{aligned} \quad (14)$$

We begin the application of the NU method [46-50] by comparing Eq. (13) with the hypergeometric differential equation

$$\psi''_n(r) + \frac{\tilde{\tau}(r)}{\sigma(r)} \psi'_n(r) + \frac{\tilde{\sigma}(r)}{\sigma^2(r)} \psi_n(r) = 0, \quad (15)$$

where

$$\psi_n(r) = \phi(r) y_n(r), \quad (16)$$

to identify the parameters,

$$\tilde{\tau}(z) = 1 + z, \quad \sigma(z) = z(1+z), \quad \tilde{\sigma}(z) = -a_2 z^2 + a_1 z - a_0^2, \quad (17)$$

and further calculate the function  $\pi(z)$  as

$$\begin{aligned} \pi(z) &= \frac{1}{2} [\sigma'(r) - \tilde{\tau}(r)] \pm \sqrt{\frac{1}{4} [\sigma'(r) - \tilde{\tau}(r)]^2 - \tilde{\sigma}(r) + k\sigma(r)} \\ &= \frac{z}{2} \pm \frac{1}{2} \sqrt{[1 + 4(a_2 + k)] z^2 + 4(k - a_1) z + 4a_0^2}. \end{aligned} \quad (18)$$

Now we also seek for a physical value of  $k$  that makes the discriminant of the expression under square root in Eq. (18) to be zero, that is

$$\begin{aligned} k &= a_1 + 2a_0^2 \pm 2a_0 q, \\ q &= \sqrt{1 + \frac{\kappa(\kappa+1) D_2}{\alpha^2 r_e^2} + \frac{4V_1 B_s}{\alpha^2}}. \end{aligned} \quad (19)$$

Upon the substitution of the value of  $k$  into Eq. (18), we obtain the following convenient solutions:

$$\pi(z) = -a_0 - \frac{1}{2}(2a_0 + q - 1)z, \quad (20)$$

and

$$k = a_1 + 2a_0^2 + a_0q. \quad (21)$$

With regard to Eqs. (17) and (20), we can calculate the function  $\tau(z) = \tilde{\tau}(z) + 2\pi(z)$ , taking into consideration the bound state condition which has to be established when  $\tau'(z) < 0$ , as

$$\tau(z) = 1 - 2a_0 - (2a_0 + q - 2)z, \quad \tau'(z) = -(2a_0 + q - 2) < 0, \quad (22)$$

where prime denotes the derivative with respect to  $z$ . According to the method [46-50], in order to find the energy equation from which one calculates the energy eigenvalues, we need to find the values of the parameters:  $\lambda = k + \pi'(s)$  and  $\lambda = \lambda_n = -n\tau'(s) - \frac{1}{2}n(n-1)\sigma''(s)$ ,  $n = 0, 1, 2, \dots$ , as

$$\lambda = \frac{1}{2} + a_1 + 2a_0^2 - a_0 + \left(a_0 - \frac{1}{2}\right)q, \quad (23)$$

and

$$\lambda_n = -n^2 - n + n(2a_0 + q), \quad n = 0, 1, 2, \dots, \quad (24)$$

respectively. Using the relation  $\lambda = \lambda_n$  and the definitions of variables in Eqs. (14) and (19), we obtain the transcendental energy equation of relativistic spin-1/2 particles in the presence of vector and scalar potential,

$$\begin{aligned} \frac{1}{\hbar^2 c^2} [M^2 c^4 - E_{n\kappa}^2 - (Mc^2 - E_{n\kappa}) C_-] &= -\frac{\kappa(\kappa+1)D_0}{r_e^2} - B_s V_2 \\ &+ \alpha^2 \left[ n + \frac{1}{2} - \frac{q}{2} + \frac{\kappa(\kappa+1)(D_1 - D_2)/r_e^2 + 2B_s V_2}{4\alpha^2(n + \frac{1}{2} - \frac{q}{2})} \right]^2, \end{aligned} \quad (25)$$

Furthermore, in the exact spin symmetric case (i.e.,  $V = S$ ,  $\Delta = 0$ ,  $C_- \rightarrow 0$ ), we obtain the arbitrary  $l$ -wave energy equation in the KG theory with equally mixed RM-type potentials (in units of  $\hbar = c = 1$ ),

$$\begin{aligned} M^2 - E_{nl}^2 &= -\frac{l(l+1)D_0}{r_e^2} - (E_{nl} + M)V_2 \\ &+ \alpha^2 \left[ n + \delta + \frac{l(l+1)(D_1 - D_2)/r_e^2 + 2(E_{nl} + M)V_2}{4\alpha^2(n + \delta)} \right]^2, \end{aligned} \quad (26)$$

with

$$\delta = \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{l(l+1)D_2}{\alpha^2 r_e^2} + \frac{4}{\alpha^2} (E_{nl} + M) V_1}, \quad (27)$$

where the quantum number  $n = 0, 1, 2, \dots$ , and the orbital quantum number  $l = 0, 1, 2, \dots$ . Actually, the above expression resembles Eq. (13) reported in [54] when  $l = 0$  ( $s$ -wave case). Now, we are going to find the corresponding wave functions for the present potential model. Firstly, we calculate the weight function defined as

$$\rho(z) = \frac{1}{\sigma(z)} \exp \left( \int \frac{\tau(z)}{\sigma(z)} dz \right) = z^{-2a_0} (1+z)^{-q}, \quad (28)$$

and the first part of the wave function in Eq. (16) as

$$\phi(z) = \exp \left( \int \frac{\pi(z)}{\sigma(z)} dz \right) = z^{-a_0} (1+z)^{\frac{1}{2}(1-q)}. \quad (29)$$

Hence, the second part of the wave function in relation (16) can be obtained by means of the so called Rodrigues representation

$$\begin{aligned} y_n(z) &= \frac{K_n}{\rho(r)} \frac{d^n}{dr^n} [\sigma^n(r) \rho(r)] = K_n z^{2a_0} (1+z)^q \frac{d^n}{dz^n} [z^{n-2a_0} (1+z)^{n-q}] \\ &\sim P_n^{(-2a_0, -q)}(1+2z), \quad z \in [0, 1], \end{aligned} \quad (30)$$

where the Jacobi polynomials  $P_n^{(\mu, \nu)}(x)$  are defined for  $Re(\nu) > -1$  and  $\Re(\mu) > -1$  for the argument  $x \in [-1, +1]$  and  $K_n$  is the normalization constant. By using  $F_{n\kappa}(z) = \phi(z)y_n(z)$ , in this way we may write the upper-spinor wave function in the fashion

$$\begin{aligned} F_{n\kappa}(r) &= K_{n\kappa} (\exp(-2\alpha r))^{-a_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q)} P_n^{(-2a_0, -q)}(1 + 2\exp(-2\alpha r)) \\ &= N_{n\kappa} (\exp(-2\alpha r))^{-a_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q)} {}_2F_1(-n, n+1-2a_0-q; -2a_0+1; \exp(-2\alpha r)), \end{aligned} \quad (31)$$

where  $a_0 > 0$ ,  $q > -1$ . The calculated normalization constants  $K_{n\kappa}$  for the upper-spinor component are

$$N_{n\kappa} = \left[ \frac{\Gamma(-q+2)\Gamma(-2a_0+1)}{2\alpha\Gamma(n)} \sum_{m=0}^{\infty} \frac{(-1)^m (n-2a_0+1-q)_m \Gamma(n+m)}{m! (m-2a_0)! \Gamma(m-2a_0-q+2)} f_{n\kappa} \right]^{-1/2}, \quad (32)$$

with

$$f_{n\kappa} = {}_3F_2(-2a_0+m, -n, n+1-2a_0-q; m-2a_0-q+2; 1-2a_0; 1). \quad (33)$$



In addition, the corresponding lower component  $G_{n\kappa}(r)$  can be obtained as follows

$$\begin{aligned}
G_{n\kappa}(r) = & c_{n\kappa} \frac{(\exp(-2\alpha r))^{-a_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q)}}{(Mc^2 + E_{n\kappa} - C_-)} \left[ -2\alpha a_0 - \frac{\alpha(1-q)\exp(-2\alpha r)}{(1 + \exp(-2\alpha r))} + \frac{\kappa}{r} \right] \\
& \times {}_2F_1(-n, n - a_0 - q + 1; -2a_0 + 1; \exp(-2\alpha r)) \\
& + c_{n\kappa} \left[ \frac{2\alpha n [n - 2a_0 - q + 1] (\exp(-2\alpha r))^{-a_0+1} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q)}}{(2a_0 + 1)(Mc^2 + E_{n\kappa} - C_-)} \right] \\
& \times {}_2F_1(-n + 1; n - 2a_0 - q + 2; -2a_0 + 2; \exp(-2\alpha r)), \quad a_0 > 0,
\end{aligned} \tag{34}$$

where  $E_{n\kappa} \neq -Mc^2$  for exact spin symmetry. Here, note that the hypergeometric series  ${}_2F_1(-n, n - 2a_0 - q + 1; -2a_0 + 1; \exp(-2\alpha r))$  terminates for  $n = 0$  and thus converge for all values of real parameters  $q > 0$  and  $a_0 > 0$ .

## B. Pspin symmetric solution

In the same way as before, this time taking the difference potential in Eq. (7) as

$$\Delta = V(r) = -4V_1 \frac{\exp(-2\alpha r)}{(1 + \exp(-2\alpha r))^2} + V_2 \left( \frac{1 - \exp(-2\alpha r)}{1 + \exp(-2\alpha r)} \right), \tag{35}$$

and in terms of new variable  $z(r) = \exp(-2\alpha r)$ , leads us to obtain a Schrödinger-like equation for the lower-spinor component  $G_{n\kappa}(r)$ ,

$$G_{n\kappa}''(z) + \frac{(1+z)}{z(1+z)} G_{n\kappa}'(z) + \frac{(-b_2 z^2 + b_1 z - b_0^2)}{z^2(1+z)^2} G_{n\kappa}(z) = 0, \tag{36}$$

where the parameters  $b_j$  ( $j = 0, 1, 2$ ) are defined by

$$\begin{aligned}
b_0 &= \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa-1)}{r_e^2} D_0 + B_{ps} V_2 + A_{ps}^2} > 0, \\
b_1 &= \frac{1}{4\alpha^2} \left( \frac{\kappa(\kappa-1)}{r_e^2} (D_1 - 2D_0) + 4B_{ps} V_1 - 2A_{ps}^2 \right), \\
b_2 &= \frac{1}{4\alpha^2} \left( \frac{\kappa(\kappa-1)}{r_e^2} (D_0 - D_1 + D_2) + A_{ps}^2 - B_{ps} V_2 \right).
\end{aligned} \tag{37}$$

To avoid repetition in the solution of Eq. (36), a first inspection for the relationship between the present set of parameters  $(b_0, b_1, b_2)$  and the previous set  $(a_0, a_1, a_2)$  tells us that the negative energy solution for pseudospin symmetry such that  $\Sigma = C_+ = C_\Sigma$  can be obtained

directly from those of the positive energy solution above for spin symmetry by performing the changes [12,20]:

$$F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), \quad V(r) \rightarrow -V(r) \text{ (or } V_1 \rightarrow -V_1, \quad V_2 \rightarrow -V_2 \text{ )}, \quad E_{n\kappa} \rightarrow -E_{n\kappa} \text{ and } C_- \rightarrow -C_+. \quad (38)$$

Considering the previous results in Eq. (25) and applying the above transformations, we finally arrive at the pspin symmetric energy equation

$$\begin{aligned} [M^2 c^4 - E_{n\kappa}^2 + (Mc^2 + E_{n\kappa}) C_+] &= -\frac{\hbar^2 c^2 \kappa (\kappa - 1) D_0}{r_e^2} + (Mc^2 - E_{n\kappa} + C_+) V_2 \\ &+ \frac{\hbar^2 c^2 \alpha^2}{4} \left[ 2n + 1 - p + \frac{\hbar^2 c^2 \kappa (\kappa - 1) (D_1 - D_2) / r_e^2 - 2 (Mc^2 - E_{n\kappa} + C_+) V_2}{\hbar^2 c^2 \alpha^2 (2n + 1 - p)} \right]^2, \end{aligned} \quad (39)$$

where

$$p = \sqrt{1 + \frac{\kappa (\kappa - 1) D_2}{\alpha^2 r_e^2} - \frac{4V_1 B_{ps}}{\alpha^2}}. \quad (40)$$

Again, the radial lower-spinor wave function in Eq. (31) becomes

$$\begin{aligned} G_{n\kappa}(r) &= d_{n\kappa} (\exp(-2\alpha r))^{-b_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-p)} P_n^{(-2b_0, -p)}(1 + 2\exp(-2\alpha r)). \\ &= d_{n\kappa} (\exp(-2\alpha r))^{-b_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-p)} {}_2F_1(-n, n + 1 - 2b_0 - p + 1; -2b_0 + 1; \exp(-2\alpha r)), \end{aligned} \quad (41)$$

which satisfies the restriction condition for the bound states, *i.e.*,  $p > 0$  and  $b_0 > 0$  and the normalization constants is

$$d_{n\kappa} = \left[ \frac{\Gamma(-p+2)\Gamma(-2b_0+1)}{2\alpha\Gamma(n)} \sum_{m=0}^{\infty} \frac{(-1)^m (n+1-2b_0-p)_m \Gamma(n+m)}{m! (m-2b_0)! \Gamma(m-2b_0-p+2)} g_{n\kappa} \right]^{-1/2}, \quad (42)$$

with

$$g_{n\kappa} = {}_3F_2(-2b_0+m, -n, n+1-2b_0-p; m-2b_0-p+2; 1-2b_0; 1). \quad (43)$$

### III. APPLICATIONS TO SOME PHYSICAL POTENTIAL MODELS

We adopt the following two physical potential cases that belong to the general potential model been introduced in Eq. (1).

### A. The reflectionless-type potential

The reflectionless-type potential is the special case of the symmetrical double-well potential offered by Büyükkılıç *et al* [55] to describe the vibration of polyatomic molecules. This can be achieved when the coefficient of  $\tanh \alpha r$  becomes zero. So, it takes the form

$$V(r) = -a^2 \sec h^2 \alpha r, \quad a^2 = \lambda(\lambda + 1)/2, \quad \lambda = 1, 2, 3, \dots \quad (44)$$

The potential is plotted in Fig. 2 for three different values  $\lambda = 1, 2$  and 3. It follows that the energy equation in Eq. (25) becomes

$$M^2 - E_{n\kappa}^2 - C_- (M - E_{n\kappa}) = -\frac{\kappa(\kappa + 1) D_0}{r_e^2} + \frac{\alpha^2}{4} \left[ 2n + 1 - q_0 + \frac{\kappa(\kappa + 1)(D_1 - D_2)/r_e^2}{\alpha^2(2n + 1 - q_0)} \right]^2, \quad (45)$$

and the upper-spinor wave functions from Eq. (31) turns to be

$$F_{n\kappa}(r) = N_{n\kappa} (\exp(-2\alpha r))^{-s_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q_0)} P_n^{(-2s_0, -q_0)}(1 + 2\exp(-2\alpha r)), \quad (46)$$

where

$$s_0 = \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa + 1)}{r_e^2} D_0 + A_s^2} > 0, \quad q_0 = \sqrt{1 + \frac{\kappa(\kappa + 1) D_2}{\alpha^2 r_e^2} + \frac{4a^2 B_s}{\alpha^2}}. \quad (47)$$

It is worth noting that the results given above in Eq. (45) and (46) are identical to those ones of Ref. [27] for  $s$ -wave case ( $\kappa = -1$ ). In the presence of the pspin case, the energy spectrum becomes

$$M^2 - E_{n\kappa}^2 + C_+ (M + E_{n\kappa}) = -\frac{\kappa(\kappa - 1) D_0}{r_e^2} + \frac{\alpha^2}{4} \left[ 2n + 1 - p_0 + \frac{\kappa(\kappa - 1)(D_1 - D_2)/r_e^2}{\alpha^2(2n + 1 - p_0)} \right]^2, \quad (48)$$

and the lower spinor component of pseudospin symmetric wave function

$$G_{n\kappa}(r) = \tilde{N}_{n\kappa} (\exp(-2\alpha r))^{-w_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-p_0)} P_n^{(-2w_0, -p_0)}(1 + 2\exp(-2\alpha r)). \quad (49)$$

where

$$w_0 = \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa - 1)}{r_e^2} D_0 + A_{ps}^2} > 0, \quad p_0 = \sqrt{1 + \frac{\kappa(\kappa - 1) D_2}{\alpha^2 r_e^2} - \frac{4a^2 B_{ps}}{\alpha^2}} > 0, \quad (50)$$

$4a^2 B_{ps}/\alpha^2 \leq 1$  for bound states when  $\kappa = 1$ .

Let us now discuss the non-relativistic limit of the energy eigenvalues and wave functions of our solution. If we take  $C_- = 0$  ( $\Delta = 0$ ) and consider the transformations  $E_{n\kappa} + M \simeq 2\mu$

and  $E_{n\kappa} - M \simeq E_{nl}$  [52,53], we would have the following expression for the energy equation (45) and wave functions (46) (in  $\hbar = c = 1$ )

$$E_{nl} = \frac{l(l+1)D_0}{2\mu r_e^2} - \frac{\alpha^2}{2\mu} \left[ n + \frac{1}{2} - \frac{1}{2}q_0 + \frac{\hbar^2 l(l+1)(D_1 - D_2)}{4r_e^2 \alpha^2 (n + \frac{1}{2} - \frac{1}{2}q_0)} \right]^2, \quad (51)$$

and the wave functions:

$$R_{nl}(r) = N_{nl} (\exp(-2\alpha r))^{-s_0} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q_0)} P_n^{(-2s_0, -q_0)}(1 + 2\exp(-2\alpha r)), \quad (52)$$

with

$$s_0 = \frac{1}{2\alpha} \sqrt{\frac{l(l+1)}{r_e^2} D_0 - \frac{2\mu}{\hbar^2} E_{nl}} > 0, \quad q_0 = \sqrt{1 + \frac{l(l+1)D_2}{\alpha^2 r_e^2} + \frac{8\mu a^2}{\alpha^2 \hbar^2}}, \quad (53)$$

where  $E_{nl} < \frac{l(l+1)}{2\mu r_e^2} D_0$  is a condition for bound state solutions. 2.

To conclude, it is necessary to mention that the reflectionless-type potential here reminds one of the modified PT potential in the one-dimensional (1D) case [29]. However, for the present case it is in the three-dimensional (3D) case. Thus, the original symmetry is broken. The energy levels could be obtained readily.

## B. The Rosen-Morse potential

The standard RM potential was given by Rosen and Morse in Ref. [45] useful to describe interatomic interaction of the linear molecules and helpful for discussing polyatomic vibrational energies. As example of its application to the vibrational states of the  $NH_3$  molecule. This can be achieved when

$$V(r) = -a(a + \alpha) \sec h^2 \alpha r + 2b \tanh \alpha r, \quad (54)$$

where  $a$  and  $b$  are real dimensionless parameters. In Fig. 3, we plot this potential for three various sets of parameter values. It follows that from Eqs. (25) and (31), the spin symmetry energy spectrum for the RM well is

$$M^2 - E_{n\kappa}^2 - C_- (M - E_{n\kappa}) = -\frac{\kappa(\kappa+1)D_0}{r_e^2} - 2b(M + E_{n\kappa} - C_-) + \frac{\alpha^2}{4} \left[ 2n + 1 - q_1 + \frac{\kappa(\kappa+1)(D_1 - D_2)/r_e^2 + 4b(M + E_{n\kappa} - C_-)}{\alpha^2(2n + 1 - q_1)} \right]^2, \quad (55)$$

and the upper spinor component  $F_{n\kappa}(r)$  of the wave functions as

$$F_{n\kappa}(r) = N_{n\kappa} (\exp(-2\alpha r))^{-s_1} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q_1)} P_n^{(-2s_1, -q_1)}(1 + 2 \exp(-2\alpha r)), \quad (56)$$

respectively, where

$$s_1 = \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa+1)}{r_e^2} D_0 + 2bB_s + A_s^2} > 0, \quad q_1 = \sqrt{1 + \frac{\kappa(\kappa+1) D_2}{\alpha^2 r_e^2} + \frac{4a(a+\alpha)B_s}{\alpha^2}}. \quad (57)$$

Overmore, in the presence of the pspin symmetry, the energy spectrum for the RM well is

$$\begin{aligned} M^2 - E_{n\kappa}^2 + C_+ (M + E_{n\kappa}) = & -\frac{\kappa(\kappa-1) D_0}{r_e^2} + 2b(M - E_{n\kappa} + C_+) \\ & + \frac{\alpha^2}{4} \left[ 2n + 1 - p_1 + \frac{\kappa(\kappa-1)(D_1 - D_2)/r_e^2 - 4b(M - E_{n\kappa} + C_+)}{\alpha^2(2n + 1 - p_1)} \right]^2, \end{aligned} \quad (58)$$

and the lower-spinor wave function is

$$G_{n\kappa}(r) = d_{n\kappa} (\exp(-2\alpha r))^{-w_1} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-p_1)} P_n^{(-2w_1, -p_1)}(1 + 2 \exp(-2\alpha r)), \quad (59)$$

with

$$w_1 = \frac{1}{2\alpha} \sqrt{\frac{\kappa(\kappa-1)}{r_e^2} D_0 + 2bB_{ps} + A_{ps}^2} > 0, \quad p_1 = \sqrt{1 + \frac{\kappa(\kappa-1) D_2}{\alpha^2 r_e^2} - \frac{4a(a+\alpha)B_{ps}}{\alpha^2}}, \quad (60)$$

where  $4a(a+\alpha)B_{ps}/\alpha^2 \leq 1$  when  $\kappa = 1$ . Let us now discuss the non-relativistic limit of the energy eigenvalues and wave functions of our solution. If we take  $C_- = 0$  ( $\Delta = 0$ ) and consider the nonrelativistic limits [52,53], we would have the following expression for the energy equation (55) and the upper spinor component of the wave functions (56) (in units  $\hbar = c = 1$ )

$$\begin{aligned} E_{nl} = & \frac{l(l+1) D_0}{2\mu r_e^2} + 2b \\ & - \frac{\alpha^2}{2\mu} \left[ n + \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{l(l+1) D_2}{\alpha^2 r_e^2} + \frac{8\mu a(a+\alpha)}{\alpha^2}} + \frac{l(l+1)(D_1 - D_2)/r_e^2 + 8\mu b}{\alpha^2 \left( n + \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{l(l+1) D_2}{\alpha^2 r_e^2} + \frac{8\mu a(a+\alpha)}{\alpha^2}} \right)} \right]^2, \end{aligned} \quad (61)$$

and

$$R_{nl}(r) = N_{nl} (\exp(-2\alpha r))^{-s_1} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-q_1)} P_n^{(-2s_1, -q_1)}(1 + 2 \exp(-2\alpha r)), \quad (62)$$

respectively, where

$$s_1 = \frac{1}{2\alpha} \sqrt{\frac{l(l+1)}{r_e^2} D_0 + 4\mu b - 2\mu E_{nl}} > 0, \quad q_1 = \sqrt{1 + \frac{l(l+1) D_2}{\alpha^2 r_e^2} + \frac{8\mu a(a+\alpha)}{\alpha^2}}. \quad (63)$$

and  $N_{nl}$  is the normalization constant.

To conclude, it is necessary to mention that the RM potential was studied by using the proper quantization rule in Ref. [33].

#### IV. DISCUSSIONS

We study two special cases of the energy eigenvalues given by Eqs. (25) and (39) for the spin and pspin symmetry, respectively.

(I) The  $s$ -wave spin symmetric case ( $\kappa = -1$ ,  $l = 0$ ) (in units  $\hbar = c = 1$ ). For the reflectionless-type potential, we have

$$M^2 - E_{n,-1}^2 - (M - E_{n,-1}) C_- = \alpha^2 \left[ n + \frac{1}{2} - \frac{1}{2} \sqrt{1 + \frac{4a^2}{\alpha^2} (M + E_{n,-1} - C_-)} \right]^2, \quad (64)$$

and

$$F_{n,-1}(r) = c_{n,-1} (\exp(-2\alpha r))^{-A_{-1}/2\alpha} (1 + \exp(-2\alpha r))^{\frac{1}{2} \left( 1 - \sqrt{1 + \frac{4a^2}{\alpha^2} (M + E_{n,-1} - C_-)} \right)} \\ \times P_n \left( -A_{-1}/\alpha, -\sqrt{1 + \frac{4a^2}{\alpha^2} (M + E_{n,-1} - C_-)} \right) (1 + 2 \exp(-2\alpha r)) \quad (65)$$

where  $A_{-1}^2 = M^2 - E_{n,-1}^2 - (M - E_{n,-1}) C_-$ . The  $\tilde{s}$ -wave pspin symmetric case ( $\kappa = 1$ ,  $\tilde{l} = 0$ ):

$$M^2 - E_{n,1}^2 + C_+ (M + E_{n,1}) = \alpha^2 \left[ n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4a^2}{\alpha^2} (M - E_{n,1} + C_+)} \right]^2, \quad (66)$$

where  $\frac{4a^2}{\alpha^2} (M - E_{n,1} + C_+) \leq 1$  and the lower spinor component of pspin symmetric wave function

$$G_{n,1}(r) = d_{n,1} (\exp(-2\alpha r))^{-A_1/2\alpha} (1 + \exp(-2\alpha r))^{\frac{1}{2} \left( 1 - \sqrt{1 - \frac{4a^2}{\alpha^2} (M - E_{n,1} + C_+)} \right)} \\ \times P_n \left( -A_1/\alpha, -\sqrt{1 - \frac{4a^2}{\alpha^2} (M - E_{n,1} + C_+)} \right) (1 + 2 \exp(-2\alpha r)). \quad (67)$$

where  $A_1^2 = M^2 - E_{n,1}^2 + (M + E_{n,1}) C_+$ . For the RM potential model (spin symmetric case), we have

$$M^2 - E_{n,-1}^2 - C_- (M - E_{n,-1}) = -2b (M + E_{n,-1} - C_-) \\ + \alpha^2 \left[ n + \frac{1}{2} - \frac{\beta_{-1}}{2} + \frac{b (M + E_{n\kappa} - C_-)}{\alpha^2 \left( n + \frac{1}{2} - \frac{\beta_{-1}}{2} \right)} \right]^2, \quad (68)$$

and the upper spinor component  $F_{n\kappa}(r)$  of the wave functions as

$$F_{n,-1}(r) = N_{n,-1} (\exp(-2\alpha r))^{-\gamma_{-1}/2} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-\beta_{-1})} P_n^{(-\gamma_{-1}, -\beta_{-1})}(1 + 2\exp(-2\alpha r)),$$

where  $\beta_{-1} = \sqrt{1 + \frac{4a(a+\alpha)}{\alpha^2} (M + E_{n,-1} - C_-)}$  and  $\gamma_{-1} = \sqrt{2b(M + E_{n,-1} - C_-) + A_{-1}^2}/2\alpha$ . For the pseudospin case:

$$\begin{aligned} M^2 - E_{n,1}^2 + C_+ (M + E_{n,1}) &= 2b(M - E_{n,1} + C_+) \\ &+ \alpha^2 \left[ \left( n + \frac{1}{2} - \frac{\beta_1}{2} \right) - \frac{b(M - E_{n,1} + C_+)}{\alpha^2 \left( n + \frac{1}{2} - \frac{\beta_1}{2} \right)} \right]^2, \end{aligned} \quad (69)$$

and the lower-spinor wave functions become

$$G_{n,1}(r) = d_{n,1} (\exp(-2\alpha r))^{-\gamma_1/2} (1 + \exp(-2\alpha r))^{\frac{1}{2}(1-\beta_1)} P_n^{(-\gamma_1, -\beta_1)}(1 + 2\exp(-2\alpha r)), \quad (70)$$

where  $\beta_1 = \sqrt{1 - \frac{4a(a+\alpha)}{\alpha^2} (M - E_{n,1} + C_+)}$  and  $\gamma_1 = \sqrt{2b(M - E_{n,1} + C_+) + A_1^2}/\alpha$ .

(II) The transformation of the potential (1) into other potential forms. For a potential  $V(x)$ , when one makes the transformations:  $x \rightarrow -x$ ,  $\alpha \rightarrow i\alpha$  and  $V_2 \rightarrow iV_2$  (complex parameters), then Eq. (1) transforms into a trigonometric Rosen-Morse-type (tRM) form:

$$V(x) = -V_1 \sec^2 \alpha x + V_2 \tan \alpha x, \quad \alpha = \frac{\pi}{2a}, \quad x = [0, a], \quad (71)$$

where  $\Re(V_1) > 0$ . When  $x \rightarrow -x$  and  $i \rightarrow -i$ , if the relation  $V(-x) = V^*$  exists, the potential  $V(x)$  is said to be  $PT$ -symmetric, where  $P$  denotes parity operator (space reflection) and  $T$  denotes time reversal (see e.g., [56,57] and the references therein). This  $PT$ -symmetric potential is plotted in Figure 4 for various sets of parameters  $V_1$  and  $V_2$ . Thus the spin-symmetric energy equation ( $\kappa = -1$ ) can be obtained from Eqs. (19) and (25) as

$$\begin{aligned} M^2 - E_{n,-1}^2 - (M - E_{n,-1}) C_- &= -\alpha^2 \left( n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4V_1}{\alpha^2} (M + E_{n,-1} - C_-)} \right)^2 \\ &+ \left( \frac{V_2}{2\alpha} \right)^2 \left( \frac{M + E_{n,-1} - C_-}{n + \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4V_1}{\alpha^2} (M + E_{n,-1} - C_-)}} \right)^2, \end{aligned} \quad (72)$$

where  $4V_1 (M + E_{n,-1} - C_-) \leq \alpha^2$ .

## V. FINAL COMMENTS AND CONCLUSION

In summary, we have obtained the approximate analytic relations for the relativistic energy spectra and the corresponding upper and lower spinor wave functions in the presence of spherical scalar and vector reflectionless-type and RM potential models under the conditions of the spin and pspin symmetries. The resulting solutions of the wave functions are being expressed in terms of the generalized Jacobi polynomials or hypergeometric functions. Parametric generalization of the NU method is used. We have further used the recently introduced exponential approximation to deal with the spin-orbit centrifugal (pseudo-centrifugal) potential term. The most stringent interesting result is that the present spin (pspin) symmetric energy spectrum of the Dirac equation is noticed to be the same as the energy spectrum of the KG solution if  $V = \pm S$  (*i.e.*,  $\Sigma = \Delta = 0$ ,  $C_{\pm} = 0$ ). We point out a possible remark of this result. The conditions that originate the spin and pspin symmetries in the Dirac equation are the same that produce equivalent energy spectra of relativistic spin-1/2 and spin-0 particles in the presence of spherical vector and scalar potentials. Obviously, the relativistic solution can be reduced to its non-relativistic limit by the choice of appropriate mapping transformations. Also, in case when spin-orbit quantum number  $\kappa = \pm 1$  ( $l = \tilde{l} = 0$ ), the problem can be easily reduced to the  $s$ -wave solution. We also find that when we let  $x \rightarrow -x$ ,  $\alpha \rightarrow i\alpha$ ,  $V_2 \rightarrow iV_2$ , the RM potential (1) turns into tRM potential (71) with real energy solution. We hope that, as in the nonrelativistic case (see, for example [58]), the relativistic model under consideration can be applied in molecular physics as well as nuclear physics. We stress that the present results should be useful in studying the rotation-vibration energy spectrum of low vibrational molecules of small amplitude and spin-orbit quantum number  $\kappa$  that are not large [51,58].

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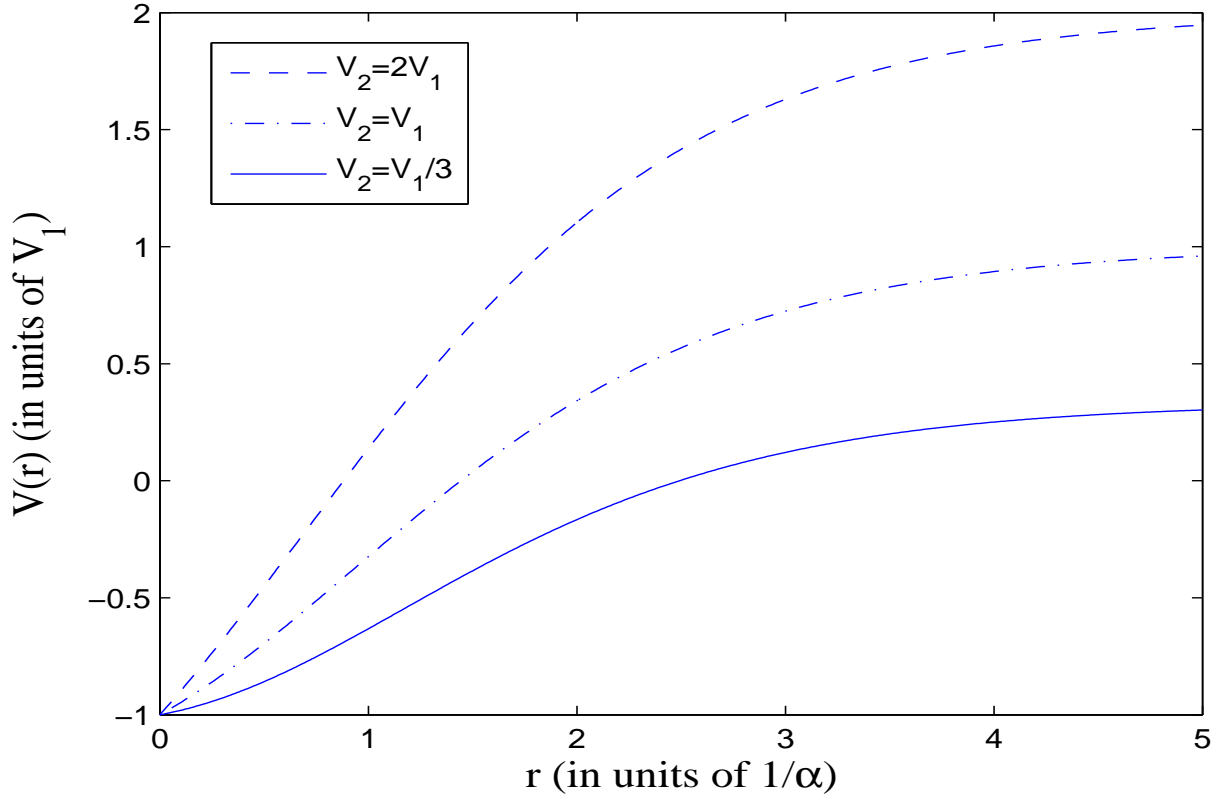


FIG. 1: A plot of the Rosen-Morse-type potential (1) for three different cases  $V_2 = 2V_1$ ,  $V_2 = V_1$  and  $V_2 = V_1/3$ .

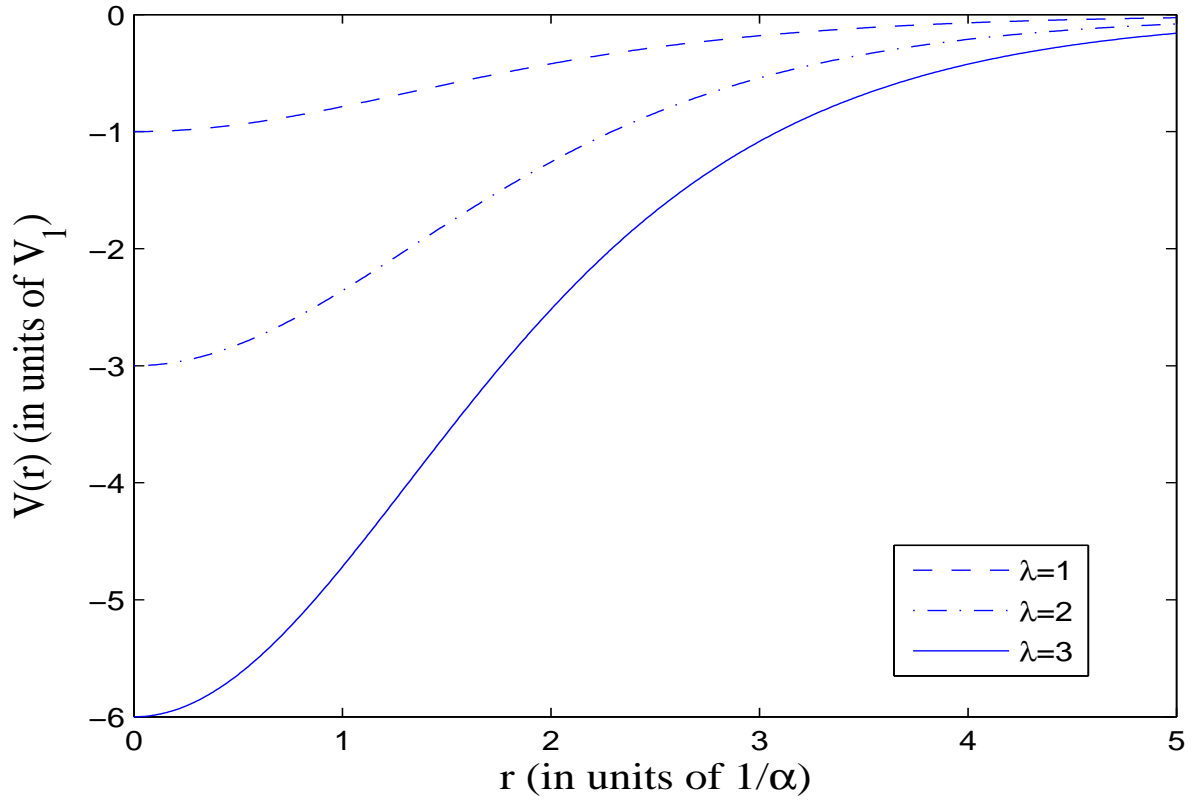


FIG. 2: A plot of the reflectionless potential (44) for three various values  $\lambda = 1, 2$ , and  $3$ .

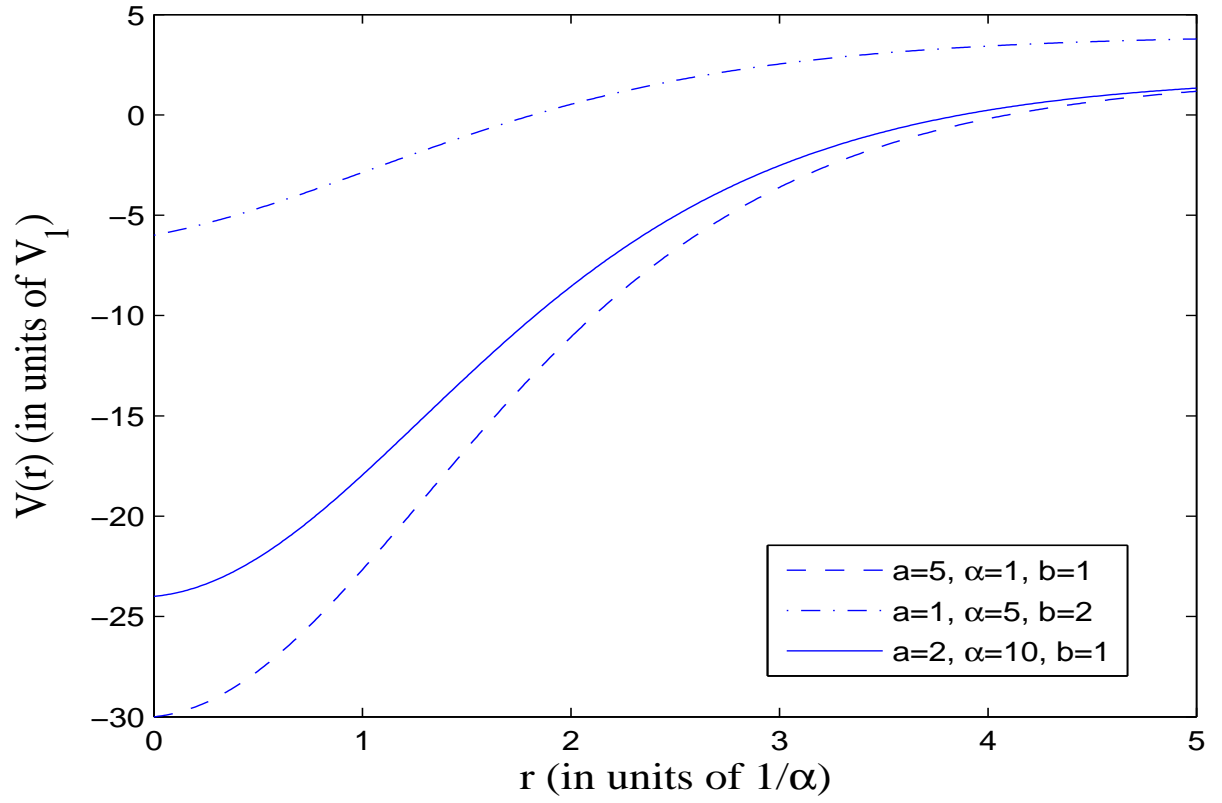


FIG. 3: A plot of the Rosen-Morse potential (54) for three different sets of parameters (i)  $a = 5$ ,  $\alpha = 1$ ,  $b = 1$ , (ii)  $a = 1$ ,  $\alpha = 5$ ,  $b = 2$ , and (iii)  $a = 2$ ,  $\alpha = 10$ ,  $b = 1$ .



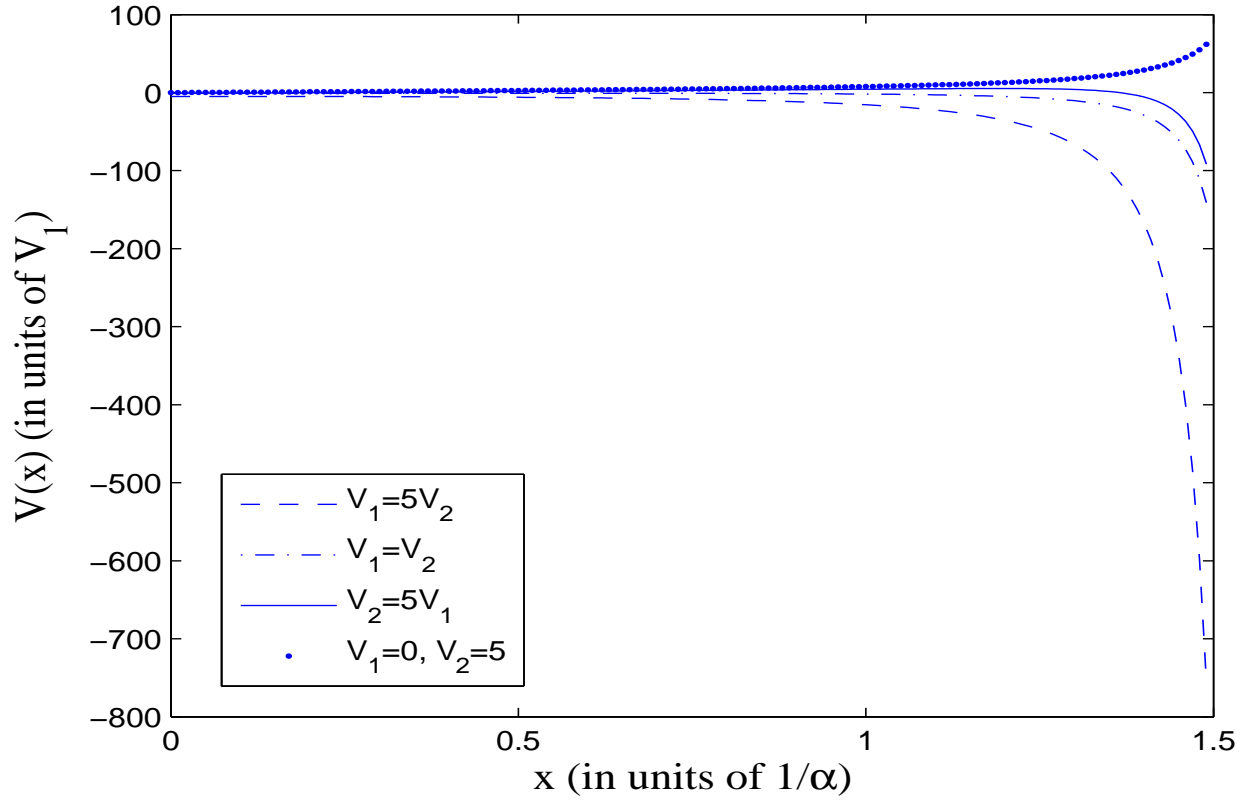


FIG. 4: Plot of the trigonometric Rosen-Morse-type potential [see Eq. (71)] for various sets of parameters.